Let $f: [a,b] \to [m,M]$ where $a,b,m,M \in \mathbb{R}$ with $a<b$, $m<M$.

Consider an ("interval-") partition P of $[a,b]$ ($P \in \text{par}[a,b]$ in symbol):

$$P := \{a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b\}$$
$$= \{I_1, I_2, \cdots, I_n\} \quad \text{(abuse of notations)}$$

where $I_i := [x_{i-1}, x_i]$ $(i=1,2,\cdots,n)$. The upper/lower Riemann sums of f w.r.t P are defined by

$$U(f;P) := \sum_{i=1}^{n} M_i \cdot \ell(I_i) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$
$$u(f;P) := \sum_{i=1}^{n} m_i \cdot \ell(I_i) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}),$$

where $M_i := \sup\{f(x) : x \in I_i\}$ $\forall\, i = 1,2,\cdots,n$.
$m_i := \inf\{\cdots\cdots\}$

The upper/lower Riemann integrals of f (on $[a,b]$) are defined by (最大下界
最小上界, respectively)

$$\overline{\int_a^b} f := \inf\{ U(f;P) : P \in par[a,b] \}$$
$$\underline{\int_a^b} f := \sup\{ u(f;P) : \cdots\cdots\cdot \}$$

(or use $(R)\overline{\int_a^b} f$, $(R)\underline{\int_a^b} f$ when we need to compare/emphasize the approaches of Riemann/Lebesgue).

HWs.

(i) $u(f;P) \leq U(f;P) \quad \forall\ P \in par[a,b]$;

(ii) $m(b-a) \leq u(f;P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f;P) \leq M(b-a)$.
$\forall\ P \in par[a,b]$.

(iii) $u(f;P) \uparrow_P$: $u(f;P) \leq u(f;P')$ $\forall P \subseteq P'$

(iv) $U(f;P) \downarrow_P$

(v) $u(f;P) \leq U(f;Q)$ $\forall P, Q \in \mathcal{P}ar[a,b]$

(Hint: Look at $\underline{\underline{P \cup Q}}$)

and $\underline{\int} f \leq \overline{\int} f$

(vi) If $P \subsetneq P'$ with $P' \setminus P = \{x'\}$, say $x_{i-1} < x' < x_i$

then $0 \leq u(f;P') - u(f;P) \leq (M-m)\|P\|$

$m_i' \ell(I_i') + m_i'' \ell(I_i'') - m_i \ell(I_i)$

where $\|P\| := \max\{\ell(I_i) : i = 1, 2, \cdots n\}$

& $P = \{I_i : i = 1, 2, \cdots, n\} = \{x_i : i = 0, 1, \cdots n\}$

(i.e. $\|P\| := \max\{x_i - x_{i-1} : i = 1, 2, \cdots n\}$)

(vii) If $P \subseteq P'$ with $\#(P' \setminus P) = N \ (\in \mathbb{N})$

then $0 \leq u(f;P') - u(f;P) \leq N(M-m)\|P\|$

&

$0 \leq U(f;P) - U(f;P') \leq N_\varepsilon (M-m)\|P\|$

$P \cup P_\varepsilon$.

Hint for (vi). $I_i = [x_{i-1}, x_i] = \underbrace{[x_{i-1}, x']}_{I_i'} \cup \underbrace{[x', x_i]}_{I_i''}$

$m_i := \inf\{f(x) : x \in I_i\}$
$m_i' := \inf\{f(x) : x \in I_i'\}$
$m_i'' := \inf\{f(x) : x \in I_i''\}$

Then

$$\begin{aligned} u(f;P') - u(f;P) &= m_i' \, l(I_i') + m_i'' \, l(I_i'') - m_i \, l(I_i) \\ &\leq M\big(l(I_i') + l(I_i'')\big) - m\, l(I_i) \\ &= M\, l(I_i) - m\, l(I_i) = (M-m)\, l(I_i) \\ &\leq (M-m)\|P\| \end{aligned}$$

$/\!/ \ell$

(viii) $\underline{\int} f := \sup_{P \in \mathcal{P}ar[a,b]} u(f;P) = \lim_{P \in \mathcal{P}ar[a,b]} u(f;P)$

and $\overline{\int} f := \inf_{P \in \mathcal{P}ar[a,b]} U(f;P) = \lim_{P} U(f;P)$

(by definition)

Hint: For the 2nd line, we have to show that:
$\forall \varepsilon > 0, \exists P_\varepsilon \in \mathcal{P}ar[a,b]$ s.t.

(*) $|\overline{\int} f - U(f;P)| < \varepsilon, \ \forall$ partition $P \supseteq P_\varepsilon$ finer.

To do this, let $\varepsilon > 0$. Then, by def, $\exists P_\varepsilon \in \mathrm{Par}[a,b]$ s.t.

$$U(f; P_\varepsilon) < \overline{\int} f + \varepsilon.$$

& def of $\overline{\int} f$

Then, $\forall$ $\mathrm{Par}[a,b] \ni P \supseteq P_\varepsilon$ one has, by (iv),

$$\overline{\int} f \leq U(f; P) \leq U(f; P_\varepsilon) < \overline{\int} f + \varepsilon$$

so $(*)$ holds.

$\boxed{(ix)^*}$

$$\overline{\int} f = \lim_{\|P\| \to 0} U(f; P) \quad \&$$

$$\underline{\int} f = \lim_{\|P\| \to 0} u(f; P)$$

think:

For the 1st line, let $\varepsilon > 0$. It suffices to show that $\exists\ \delta > 0$ such that

$(**)$
$$\left| U(f; P) - \overline{\int} f \right| < 2\varepsilon, \quad \forall P \in \mathrm{Par}[a,b] \text{ with } \|P\| < \delta$$

To do this, let $\varepsilon > 0$ and take P_ε, as before: $U(f; P_\varepsilon) < \overline{\int} f + \varepsilon$.

Let $N_\varepsilon := \#(P_\varepsilon)$ and let $\delta > 0$ be s.t.

$(\#)$ $$N_\varepsilon (M - m)\,\delta < \varepsilon.$$

Let $P \in \mathrm{Par}[a,b]$ with $\|P\| < \delta$. Then, by (vii), one has (applied to P, $P \cup P_\varepsilon$ in place of P, P')

$$\overline{\int} f + \varepsilon > U(f;P_\varepsilon) \overset{(iv)}{\geq} U(f;P\cup P_\varepsilon) \geq U(f;P) - \underbrace{N_\varepsilon (M-m)\cdot \|P\|}_{\delta}$$

because $\#\big((P\cup P_\varepsilon)\setminus P\big) \leq \#(P_\varepsilon) = N_\varepsilon$

$U(f;P) - N_\varepsilon(M-m)\delta$

$\vee$

$U(f;P) - \varepsilon$ by (#).

$$\therefore \overline{\int} f + 2\varepsilon > U(f;P) \overset{\text{by def of } \overline{\int} f}{\geq} \overline{\int} f \quad \&$$

$$\left| U(f;P) - \overline{\int} f \right| < 2\varepsilon,$$

valid for all P with $\|P\| < \delta$.

$\boxed{(X)^*}$ Let $\mathcal{B}[a,b]$ consist of all bounded real-valued function on $[a,b]$. Then, $\forall f, g \in \mathcal{B}[a,b]$,

$$\overline{\int}(f+g) \leq \overline{\int} f + \overline{\int} g \qquad (1)$$

$$\underline{\int}(f+g) \geq \underline{\int} f + \underline{\int} g \qquad (2)$$

$$-\underline{\int} f = \overline{\int}(-f) \quad \& \quad \overline{\int}(cf) = c\overline{\int} f,\ \underline{\int}(cf) = c\underline{\int} f \ \forall c \geq 0.$$

If $\underline{\int} f = \overline{\int} f$ then say $f \in \mathcal{R}[a,b]$ & $\int f := \underline{\int} f = \overline{\int} f$. By definitions and (X), one has: $f \mapsto \int f$ is linear, monotone and $\int f = \int f'$ if $f = f'$ on $[a,b]\setminus A$ with $\#(A) \in \mathbb{N}$.

Hint for (x): For (1), let $\varepsilon > 0$. Then $\exists$ P, Q $\in$ par[a,b] s.t.

$$U(f;P) < \overline{\int} f + \varepsilon$$
$$U(g;P') < \overline{\int} g + \varepsilon$$

and so

$$U(f;P\cup P') \leq U(f;P) < \overline{\int} f + \varepsilon \quad \&$$
$$U(g;P\cup P') < \overline{\int} g + \varepsilon$$

Consequently

pl. check via Δ-ineq

$$\overline{\int}(f+g) \leq U(f+g; P\cup P') \leq U(f;P\cup P') + U(g;P\cup P') < \overline{\int} f + \overline{\int} g + 2\varepsilon$$

and $\overline{\int}(f+g) \leq \overline{\int} f + \overline{\int} g$ as $\varepsilon > 0$ arbitrary.

Below is a revisit of

$$\overline{\int} f = \lim_{P \in par[a,b]} U(f,P) = \lim_{\|P\| \to 0} U(f,P)$$

&

$$\underline{\int} f = \lim_{P \in par[a,b]} u(f,P) = \lim_{\|P\| \to 0} u(f,P).$$

with the definitions of limits:

$\ell = \lim_{P \in par[a,b]} u(f,P)$ means: $\forall \varepsilon > 0$, $\exists$ $P_\varepsilon \in par[a,b]$ s.t. if $P \in par[a,b]$ with $P \supseteq P_\varepsilon$ then $|u(f,P) - \ell| < \varepsilon$.

$\ell = \lim_{\|P\| \to 0} u(f, P)$ means: $\forall \varepsilon > 0$

$\exists \delta > 0$ s.t. if $P \in \text{par}[a, b]$ with

$\|P\| < \delta$ then

$$|u(f, P) - \ell| < \varepsilon.$$

Let $\varepsilon > 0$. Since $\overline{\int} f = \inf\{ U(f,P) : P \in \text{par}[a,b] \}$

= 最大下界 ,

$\overline{\int} f + \varepsilon > U(f, P_\varepsilon)$ for some $P_\varepsilon \in \text{par}[a,b]$

so $\overline{\int} f + \varepsilon > U(f, P_\varepsilon) \geqslant U(f,P) \ \forall\ P \supseteq P_\varepsilon$

$\geqslant \overline{\int} f$

showing that $\lim\limits_{P} U(f,P) = \overline{\int} f$ by def of $\lim\limits_{P}$

Further, let $N_\varepsilon := \#(\text{partition pts of } P_\varepsilon)$ and

let $\delta := \dfrac{\varepsilon}{N_\varepsilon \cdot (M-m)} (>0)$, where $f : [a,b] \to [m,M] \subseteq \mathbb{R}$
(with $m < M$ wlog)

Then, $\forall\ P \in \text{par}[a,b]$ with $\|P\| < \delta$, one has

partition interval 最大長度

$$\overline{\int} f + \varepsilon > U(f, P_\varepsilon) \geqslant U(f, P \cup P_\varepsilon)$$

$$> U(P) - N_\varepsilon (M-m) \|P\|$$ (see Lemma below applied to $P \cup P_\varepsilon$ in place of P')

$$> U(P) - N_\varepsilon (M-m)\delta$$

$$= U(P) - \varepsilon, \text{ so}$$

$$\overline{\int} f + 2\varepsilon > U(P) \geqslant \overline{\int} f,$$

showing that

$$\left| U(P) - \bar{\int} f \right| < 2\varepsilon,$$ valid

for all $P \in par[a,b]$ with $\|P\| < \delta$.

$$\therefore \lim_{\|P\| \to 0} U(P) = \bar{\int} f$$

Lemma. Let partitions $P' \supseteq P$ s.t. $\#(P' \setminus P) = N \in \mathbb{N}$. Then

$$0 \leq U(P) - U(P') \leq N(M-m)\|P\|$$

Proof of Lemma. By M.I., it suffices to show for the case when $N = 1$. To do this, let us repeat the standard notation for partition P :

$$P = \{ a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \}$$

$$= \{ I_1, I_2, \cdots, I_n \},$$ abuse of notations

Let $P' = P \cup \{x'\}$, say

$$x' \in \operatorname{int}(I_j) \qquad x_{j-1} < x' < x_j.$$

so $I_j = I_j' \cup I_j''$ where $I_j' = [x_{j-1}, x']$

$I_j'' = [x', x_j]$

$$M_j' := \max\{ f(x) : x \in I_j' \} \qquad M_j = \max\{ f(x) : x \in I_j \}$$

$$M_j'' := \max\{ f(x) : x \in I_j'' \}$$

Note that

interval 長度

$$U(f, P) - U(f, P \cup \{x'\})$$

$$= M_j \cdot \ell(I_j) - \left(M_j' \cdot \ell(I_j') + M_j'' \cdot \ell(I_j'') \right)$$

$$\leq M \cdot \ell(I_j) - \left(m \cdot \ell(I_j') + m \cdot \ell(I_j'') \right)$$

$$= (M - m)\, \ell(I_j) \leq (M - m)\, \|P\|,$$

proving the lemma for $N = 1$.

Remark. Similarly for $\underline{\int} f$ and $u(f,P)$:

$$0 \leq u(f, P \cup \{x'\}) - u(f,P) \qquad (x_{j-1} < x' < x_j)$$

$$= \left(m_j' \, \ell(I_j') + m_j'' \, \ell(I_j'')\right) - m_j \, \ell(I_j)$$

$$= \left(m_j' \, \ell(I_j') + m_j'' \, \ell(I_j'')\right) - \left(m_j \, \ell(I_j') + m_j \, \ell(I_j'')\right)$$

$$\leq (M-m) \, \ell(I_j) \leq (M-m) \|P\| .$$

Vibration Functions

$$f_\delta(x) := \inf\{f(u) : u \in V_\delta(x)\} \qquad (\delta > 0)$$
$$f^\delta(x) := \sup\{f(u) : u \in V_\delta(x)\}$$

(so $f_\delta \leq f \leq f^\delta$).

$f_\delta \uparrow$ & $f^\delta \downarrow$ (as $\delta \downarrow$)

$\underline{f} := \sup f_\delta$ (ptwise)

$\overline{f} := \inf f^\delta$ (...).

Note. Take, iteratively, (by bisection process) a seq (P_n) of partitions of $[0,1]$ s.t.

$$P_n \subseteq P_{n+1} \quad \forall n \in \mathbb{N}$$
$$\|P_n\| \leq \frac{1}{2^n} \quad \forall n \in \mathbb{N}$$

(so $\lim_n U(f; P_n) = \overline{\int} f$, $\quad U(f; P_n) \downarrow_n$

$\lim_n u(f; P_n) = \underline{\int} f \quad u(f; P_n) \uparrow_n$)

Let $P_n = \{I_1^{(n)}, I_2^{(n)}, \cdots I_K^{(n)}\}$ and take two step functions $\varphi_n, \psi_n : [0,1] \to [m, M]$ s.t. $\varphi_n \leq f \leq \psi_n$ such that $\forall i = 1, 2, \cdots K$,

$\varphi_n = m_i$ on int$(I_i^{(n)})$ interior

$\psi_n = M_i$ on int $I_i^{(n)}$)

(say $\varphi_n = m$ at all partition points of P_n
$\psi_n = M$ — — — — — — —)

Then $\int \varphi_n = u(f; P_n)$
$\int \psi_n = U(f; P_n)$)

Thm. Let

$$X := [0,1] \setminus \left\{ \frac{k}{2^n} : n \in \mathbb{N},\ k = 0, 1, \cdots, 2^n \right\}$$

Then

$$\varphi_n \uparrow \underline{f} \text{ and } \psi_n \downarrow \bar{f} \quad \text{on } X,$$

where, $\forall x \in [0,1]$, by definitions

$$\underline{f}(x) \overset{\text{def}}{=} \sup\{ f_\delta(x) : \delta > 0 \}, \quad f_\delta(x) :\overset{\text{def}}{=} \inf\{ f(u) : u \in V_\delta(x) \cap [0,1] \}$$

and

$$\bar{f}(x) \overset{\text{def}}{=} \inf\{ f^\delta(x) : \delta > 0 \}, \quad f^\delta(x) :\overset{\text{def}}{=} \sup\{ f(u) : u \in V_\delta(x) \cap [0,1] \}$$

Indeed, let $x_0 \in X$. Then, $\forall n$, $x_0 \in \operatorname{int}(I_i^{(n)})$ for some $i \in \{1, 2, \cdots 2^n\}$); take $\delta > 0$ s.t. $V_\delta(x_0) \subseteq I_i^{(n)}$. Note then that $f^\delta(x_0) := \sup\{ f(x) : x \in V_\delta(x_0) \} \leq M_i^{(n)} = \psi_n(x_0)$ and it follows that $\bar{f}(x_0) \left(= \inf_{\gamma > 0} f^\gamma(x_0) \right) \leq f^\delta(x_0) \leq \psi_n(x_0)$. Thus we arrive at

$$\bar{f}(x_0) \leq \psi_n(x_0), \quad \forall n \in \mathbb{N}.$$

So

$$\bar{f}(x_0) \leq \inf\{ \psi_n(x_0) : n \in \mathbb{N} \} \qquad (*)$$

Conversely, $\forall \delta > 0$, $\exists$ some $n \in \mathbb{N}$ s.t. $\frac{1}{2^n} < \delta$

For this n, $x_0 \in \operatorname{int}(I_i^{(n)})$ for some i as before. For these n and i one has

$$I_i^{(n)} \subseteq V_\delta(x_0)$$

(because

$$|x - x_0| \leq l\left(I_i^{(n)}\right) = \frac{1}{2^n} < \delta \qquad \forall x \in I_i^{(n)}\Big).$$

Therefore

$$\sup\{f(x) : x \in I_i^{(n)}\} \leq \sup\{f(x) : x \in V_\delta(x_0)\}$$

i.e.

$$\psi_n(x_0) = M_i^{(n)} \leq f^\delta(x_0)$$

This implies that

$$\inf\{\psi_K(x_0) : K = 1, 2, \cdots\} \leq \psi_n(x_0) \leq f^\delta(x_0).$$

Since $\delta > 0$ is arbitrary, this implies

$$\inf\{\psi_K(x_0) : K = 1, 2, \cdots\} \leq \bar{f}(x_0)$$

(so equal by the earlier (*).